

Integral trees with given nullity

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Abstract

A graph is called integral if all eigenvalues of its adjacency matrix consist entirely of integers. We prove that for a given nullity more than 1, there are only finitely many integral trees. It is also shown that integral trees with nullity 2 and 3 are unique.

Keywords: adjacency eigenvalue, eigenvalue multiplicity, nullity, integral tree.

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1 Introduction

For a graph G , we denote by $\mathcal{V}(G)$, the vertex set of G and the *order* of G is defined as $|\mathcal{V}(G)|$. The *adjacency matrix* of G , denoted by $\mathcal{A}(G)$, has its rows and columns indexed by $\mathcal{V}(G)$ and its (u, v) -entry is 1 if the vertices u and v are adjacent and 0 otherwise. The *characteristic polynomial* of G , denoted by $\varphi(G)$, is the characteristic polynomial of $\mathcal{A}(G)$. The zeros of $\varphi(G)$ are called the *eigenvalues* of G . Note that $\mathcal{A}(G)$ is a real symmetric matrix so that all eigenvalues of G are reals. We denote the eigenvalues of G in non-increasing order as $\lambda_1(G) \geq \dots \geq \lambda_n(G)$, where $n = |\mathcal{V}(G)|$. The graph G is said to be *integral* if all eigenvalues of G are integers. The *nullity* of G is defined as the nullity of $\mathcal{A}(G)$, which is equal to the multiplicity of 0 as an eigenvalue of G .

The notion of integral graphs was first introduced in [4]. There is not much knowledge about integral graphs in the literature. Here, we are concerned with integral trees. These objects

are extremely rare and hence very difficult to find. For a long time, it was an open question whether there exist integral trees with arbitrarily large diameter [7]. Recently, this question was affirmatively answered in [2, 3], where the authors constructed integral trees for any diameter. It is well known that the tree on two vertices is the only integral tree with nullity zero [8]. Thereafter, Brouwer proved that any integral tree with nullity 1 is a subdivision of a star where the order of the star is a perfect square [1]. This result has motivated us to investigate integral trees from the ‘nullity’ point of view.

In this article, we prove that with a fixed nullity more than 1, there are only finitely many integral trees. We also characterize integral trees with nullity 2 and 3 showing that there is a unique integral tree with nullity 2 as well as a unique integral tree with nullity 3.

2 Reduced trees

We denote the path graph of order n by P_n . For a vertex v of a tree T , we say that there are k pendant P_2 at v if $T - v$ has k components P_2 . A tree T is called *reduced* if it has no pendant P_2 at each vertex. We denote the multiplicity of λ as an eigenvalue of a graph G by $\text{mult}(G; \lambda)$. We also denote the number of eigenvalues of G in the interval $(-1, 1)$ by $m(G)$. We first state the following well known fact.

Lemma 1. *Let G be a graph and $u \in \mathcal{V}(G)$ be the unique neighbor of $v \in \mathcal{V}(G)$. Then the nullities of G and $G - \{u, v\}$ are the same.*

The following well known result is immediately deduced from Lemma 1.

Corollary 2. *The size of the maximum matching in a tree of order n with nullity h is $\frac{n-h}{2}$.*

The first statement of the following theorem is called the ‘interlacing theorem’ which has a key role in spectral graph theory.

Theorem 3. *Let G be a graph of order n and let H be an induced subgraph of G of order m . Then $\lambda_{n-m+i}(G) \leq \lambda_i(H) \leq \lambda_i(G)$, for $i = 1, \dots, m$. Moreover, if G is a connected graph and $G \neq H$, then $\lambda_1(H) < \lambda_1(G)$.*

Lemma 4. *In every tree, removing each pendant P_2 does not increase the number of eigenvalues in $(-1, 1)$.*

Proof. Assume that u and v are two adjacent vertices of a tree T with degrees 1 and 2, respectively. Letting $T' = T - \{u, v\}$, we show that $m(T') \leq m(T)$. Lemma 1 yields that $\text{mult}(T - v; 0) = \text{mult}(T; 0) + 1$ and so $m(T') = m(T - v) - 1$. Applying Theorem 3 for T and $T - v$, we easily find that $m(T - v) \leq m(T) + 1$, the result follows. \square

Lemma 5. *The paths P_1 and P_2 are the only reduced trees with at most one eigenvalue in $(-1, 1)$.*

Proof. By contradiction, assume that $T \notin \{P_1, P_2\}$ is a tree with minimum possible order such that $m(T) \leq 1$. Let $v \in \mathcal{V}(T)$ be adjacent to a vertex of degree 1. Since T is reduced, the degree of v is at least 3. By Theorem 3, we have $m(T - v) \leq m(T) + 1 \leq 2$ and so one of the connected components of $T - v$ has no eigenvalue in $(-1, 1)$. By the minimality of T , this connected component must be P_2 , which is impossible since T is reduced. \square

Since P_2 is the only reduced tree with no eigenvalue in $(-1, 1)$ and $m(P_4) = 2$, using Lemmas 4 and 5, we obtain the following conclusion which generalizes a result in [8].

Corollary 6. *The tree P_2 is the only one with no eigenvalue in $(-1, 1)$.*

Theorem 7. *For any given nonnegative integer k , there are finitely many reduced trees with exactly k eigenvalues in $(-1, 1)$.*

Proof. We prove the assertion by induction on k . By Lemma 5, we may assume that $k \geq 2$. Let T be a reduced tree with $m(T) = k$. Since T is reduced, T is not a path and hence it has a vertex of degree at least 3, say v . By Theorem 3, $m(T - v) \leq m(T) + 1$. Since P_2 is not a connected component of $T - v$, every connected component of $T - v$ has at most $k - 1$ eigenvalues in $(-1, 1)$. Now we are done, by the induction hypothesis. \square

For a later use, we need the following refinement of Lemma 4.

Lemma 8. *Let T be a tree with at least one pendant P_2 at $v \in \mathcal{V}(T)$. Then increasing the number of pendant P_2 at v by one, leaves the number of eigenvalues in $(-1, 1)$ unchanged and increases the multiplicity of 1 by one.*

Proof. Suppose that T' is the resulting tree from T by adding two new vertices a and b which a is joined to both b and v . By Theorem 3, it suffices to show that $m(T) = m(T')$. Since $m(T' - a) = m(T) + 1$, we proceed to prove that $m(T' - a) = m(T') + 1$. Let $\{x_1, \dots, x_k\}$ be a basis for eigenspace \mathcal{E} of T corresponding to the eigenvalue 1. Since any vector of \mathcal{E} takes the same value on c and d , we conclude that each vector of \mathcal{E} vanishes on v , where c and d are vertices of a pendant P_2 of T at v . For $i = 1, \dots, k$, extend x_i to an eigenvector y_i of T' corresponding to the eigenvalue 1 with value 0 on $\{a, b\}$. Define the vector y_{k+1} such that $y_{k+1}(a) = y_{k+1}(b) = 1$, $y_{k+1}(c) = y_{k+1}(d) = -1$, and 0 elsewhere. Clearly, $\{y_1, \dots, y_{k+1}\}$ is a basis for the eigenspace of T' corresponding to the eigenvalue 1. This shows that the $\text{mult}(T'; 1) = \text{mult}(T' - a; 1) + 1$. By Lemma 1, $\text{mult}(T' - a; 0) = \text{mult}(T'; 0) + 1$, and applying Theorem 3 for T' and $T' - a$, we clearly obtain that $m(T' - a) = m(T') + 1$, as desired. \square

3 Finiteness of integral trees with a given nullity

Let $n \geq 1$ and r_1, \dots, r_n be nonnegative integers. Starting from one endpoint, label the vertices of P_{2n+1} with v_1, \dots, v_{2n+1} . Attach r_i new isolated vertices to v_{2i} , for $i = 1, \dots, n$. We denote the resulting tree by $C(r_1, \dots, r_n)$. Next, we attach a new vertex to each vertex of degree 1 of $C(r_1, \dots, r_n)$ and also each vertex in $\{v_3, \dots, v_{2n-1}\}$. Denote the new tree by $S(r_1, \dots, r_n)$. For instance, the trees $C(1, 3, 0, 2)$ and $S(1, 3, 0, 2)$ are depicted in Figure 1. The vertices v_2, \dots, v_{2n} are said to be *central*. We will establish that no tree of the form $S(r_1, \dots, r_n)$ with $n \geq 2$ is integral. We first recall the next lemma.

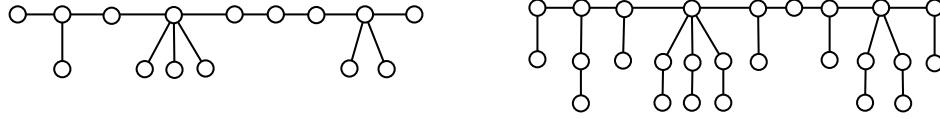


Figure 1: The trees $C(1, 3, 0, 2)$ (left) and $S(1, 3, 0, 2)$ (right)

Lemma 9. [2] *Let G be a bipartite graph with bipartition $\mathcal{V}(G) = V_1 \cup V_2$ and with k positive eigenvalues. Let G' be the graph obtained from G by joining each vertex of V_1 to r new vertices of degree 1. Then $\lambda_i^2(H) = \lambda_i^2(G) + r$, for $i = 1, \dots, k$.*

In view of Theorem 3 and Lemma 9, the proof of the following lemma is straightforward.

Lemma 10. *Let $n \geq 2$ and r_1, \dots, r_n be nonnegative integers. If $T = C(r_1, \dots, r_n)$ and $s \geq t$ are the two largest numbers among r_1, \dots, r_n , then $s + 2 < \lambda_1^2(T) < s + 4$ and $\lambda_2^2(T) \geq t$.*

Lemma 11. *Let $T = C(0, \dots, 0, 2, 0, \dots, 0)$. Then $\lambda_1^2(T) < 5$.*

Proof. By a direct calculation, we find that the largest eigenvalue of the graph Q depicted in Figure 2 is $\sqrt{5}$. As a result from [5], the largest eigenvalue decreases by subdividing an edge in the cycle of Q . By repeating this process, we obtain a graph which has T as an induced subgraph. So, using Theorem 3, the assertion follows. \square

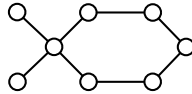


Figure 2: The graph Q .

Theorem 12. *Let $n \geq 2$ and r_1, \dots, r_n be nonnegative integers. Then $S(r_1, \dots, r_n)$ is not integral.*

Proof. Towards a contradiction, assume that $S(r_1, \dots, r_n)$ is integral. We delete all pendant vertices and obtain the tree $T = C(r_1, \dots, r_n)$. By Lemma 9, any positive eigenvalue of T is of the form $\sqrt{m^2 - 1}$, for some integer $m \geq 2$. Note that $\lambda_2(T) \geq \lambda_2(P_{2n+1}) > 0$, using Theorem 3. This implies that $\lambda_1^2(T) - \lambda_2^2(T) \geq 5$. Let $s \geq t$ be the two largest numbers among r_1, \dots, r_n . By Lemma 10, $\lambda_1^2(T) = s + 3$, which implies that $s \geq t + 2$. By Theorem 3 and Lemma 9, $\lambda_1^2(T) \leq \lambda_1^2(C(0, \dots, 0, 2, 0, \dots, 0)) + s - 2$. Now, Lemma 11 shows that $\lambda_1^2(T) < s + 3$, a contradiction. \square

Let $k \geq 1$ and T_1, T_2 be two vertex disjoint trees with specified vertices $v_1 \in \mathcal{V}(T_1)$, $v_2 \in \mathcal{V}(T_2)$. If T is the tree obtained from T_1 and k copies of T_2 by joining v_1 to each copy of v_2 , then it is well known that

$$\varphi(T) = \varphi(T_2)^{k-1}(\varphi(T_1)\varphi(T_2) - k\varphi(T_1 - v_1)\varphi(T_2 - v_2)). \quad (1)$$

Let T be a tree of order n and $k \geq 1$. For positive integers s_1, \dots, s_k and distinct vertices $v_1, \dots, v_k \in \mathcal{V}(T)$, we denote by $T(v_1, \dots, v_k; s_1, \dots, s_k)$ the resulting tree from T by attaching s_i copies of pendant P_2 at v_i , for $i = 1, \dots, k$. Assuming that $s_1 \geq \dots \geq s_k$ and applying the Courant–Weyl inequalities, we find that

$$\lambda_i(T(v_1, \dots, v_k; s_1, \dots, s_k)) \geq \sqrt{s_i + 1} + \lambda_n(T), \quad (2)$$

for $i = 1, \dots, k$. Therefore, when all values s_1, \dots, s_k go to infinity, then the i th largest eigenvalue of $T(v_1, \dots, v_k; s_1, \dots, s_k)$ is not bounded, for $i = 1, \dots, k$.

Lemma 13. *Let T be a tree, k, t be positive integers, and $v_1, \dots, v_k \in \mathcal{V}(T)$. Suppose that there exists a polynomial $f(x)$ such that for every integers $s_1, \dots, s_k \geq t$, the tree $T' = T(v_1, \dots, v_k; s_1, \dots, s_k)$ satisfies*

$$\varphi(T') = (x^2 - 1)^{s_1 + \dots + s_k - k} f(x) \prod_{i=1}^k (x^2 - \alpha_i(s_1, \dots, s_k)), \quad (3)$$

where $\alpha_i(s_1, \dots, s_k)$ is a positive-valued function in terms of s_1, \dots, s_k , for $i = 1, \dots, k$. Then $T = S(r_1, \dots, r_k)$, for some nonnegative integers r_1, \dots, r_k with v_1, \dots, v_k being the central vertices.

Proof. We prove the assertion by induction on k . Assume that $k = 1$. For simplicity, let $s = s_1$, $v = v_1$, and $\alpha(s) = \alpha_1(s_1)$. By (1), we have $\varphi(T') = (x^2 - 1)^{s-1}((x^2 - 1)\varphi(T) - sx\varphi(T - v))$. Therefore, using (3), we find that $(x^2 - 1)\varphi(T) - sx\varphi(T - v) = f(x)(x^2 - \alpha(s))$, for any integer $s \geq t$. In particular, we have

$$(x^2 - 1)\varphi(T) - tx\varphi(T - v) = f(x)(x^2 - \alpha(t)) \quad (4)$$

and

$$(x^2 - 1)\varphi(T) - (t + 1)x\varphi(T - v) = f(x)(x^2 - \alpha(t + 1)). \quad (5)$$

From (4) and (5), one obtains that $f(x) = x\varphi(T-v)(\alpha(t+1) - \alpha(t))$. By (3), it is clear that $f(x)$ is a monic polynomial implying that $\alpha(t+1) - \alpha(t) = 1$. So, $f(x) = x\varphi(T-v)$. It follows that $(x^2 - 1)\varphi(T) = x(x^2 - \mu)\varphi(T-v)$, for some positive integer μ . Thus $\text{mult}(T; 0) = \text{mult}(T-v; 0) + 1$, and so by Lemma 1, v is not adjacent to a vertex of degree 1. Consequently, T contains $S(r)$ as an induced subgraph with the central vertex v , where r is the degree of v . We know that the sum of squares of all eigenvalues of a tree of order n equals $2(n-1)$. Applying this fact to T and $T-v$, we obtain that $r = \mu - 1$. This means that $\lambda_1(T) = \lambda_1(S(r))$. By Theorem 3, $T = S(r)$, as desired.

Now assume that $k \geq 2$. By (1), we have $\varphi(T') = (x^2 - 1)^{s_k - 1}((x^2 - 1)\varphi(T'') - s_k x\varphi(T'' - v_k))$, where $T'' = T(v_1, \dots, v_{k-1}; s_1, \dots, s_{k-1})$. Hence, using (3) and setting $\rho = s_1 + \dots + s_{k-1} - k + 1$, we find that

$$(x^2 - 1)\varphi(T'') - s_k x\varphi(T'' - v_k) = (x^2 - 1)^\rho f(x) \prod_{i=1}^k (x^2 - \alpha_i(s_1, \dots, s_k)),$$

for every integers $s_1, \dots, s_k \geq t$. In particular, we have

$$(x^2 - 1)\varphi(T'') - tx\varphi(T'' - v_k) = (x^2 - 1)^\rho f(x) \prod_{i=1}^k (x^2 - \alpha_i(s_1, \dots, s_{k-1}, t)) \quad (6)$$

and

$$(x^2 - 1)\varphi(T'') - (t+1)x\varphi(T'' - v_k) = (x^2 - 1)^\rho f(x) \prod_{i=1}^k (x^2 - \alpha_i(s_1, \dots, s_{k-1}, t+1)), \quad (7)$$

for every integers $s_1, \dots, s_{k-1} \geq t$. From (6) and (7), we obtain that

$$\varphi(T'' - v_k) = (x^2 - 1)^\rho g(x) \prod_{i=1}^{k-1} (x^2 - \beta_i(s_1, \dots, s_{k-1})), \quad (8)$$

where $f(x) = xg(x)$ and $\beta_i(s_1, \dots, s_{k-1})$ is a positive-valued function in terms of s_1, \dots, s_{k-1} , for $i = 1, \dots, k-1$. By (2), (8), Lemma 8, and the induction hypothesis, every connected component of $T - v_k$ containing some of v_1, \dots, v_{k-1} is of the form $S(r_1, \dots, r_\ell)$, for some nonnegative integers r_1, \dots, r_ℓ .

Note that in our argument in the previous paragraph, v_k can be replaced with each of v_1, \dots, v_{k-1} . Now, it is straightforward to check that the assertion follows whenever $k \geq 3$. Hence, assume that $k = 2$. Since $f(x) = xg(x)$, T' has eigenvalue 0 and so Corollary 2 implies that T' and T have no perfect matching. It follows that T is of the form $S(r_1, r_2)$, for some nonnegative integers r_1 and r_2 . This completes the proof. \square

Now we are in a position to state our main theorem.

Theorem 14. *For every integer $h \geq 2$, there are finitely many integral trees with nullity h .*

Proof. By contradiction, suppose that there are infinitely many integral trees with nullity h , for some $h \geq 2$. By Theorem 7, there is a tree T with $\mathcal{V}(T) = \{v_1, \dots, v_n\}$ such that $T(v_1, \dots, v_n; s_{i1}, \dots, s_{in})$ is integral for an infinite set $\{(s_{i1}, \dots, s_{in})\}_{i \in \mathbb{N}}$ of n -tuples of nonnegative integers. If for some fixed integers j and s , the set $\{i \mid s_{ij} = s\}$ is infinite, then we replace T by $T(v_j; s)$. Repeating this operation, we may assume that there is a tree T of order n with specified vertices v_1, \dots, v_k and an infinite set $\{(s_{i1}, \dots, s_{ik})\}_{i \in \mathbb{N}}$ of k -tuples of nonnegative integers such that $s_{ij} < s_{(i+1)j}$, for $j = 1, \dots, k$, and $T_i = T(v_1, \dots, v_k; s_{i1}, \dots, s_{ik})$ is integral, for all i .

By (2), the set $\{\lambda_j(T_i) \mid i \in \mathbb{N}\}$ is not bounded, for $j = 1, \dots, k$, and the set $\{\lambda_{k+1}(T_i) \mid i \in \mathbb{N}\}$ is bounded above by $\lambda_1(T - \{v_1, \dots, v_k\})$, using Theorem 3. This clearly implies that there exists an integer i_0 such that $\lambda_j(T_i)$ is fixed, for $j = k+1, \dots, k + \frac{n-h}{2}$ and each $i \geq i_0$. Furthermore, by Lemma 8, we have $\lambda_j(T_i) = 1$, for $j = k + \frac{n-h}{2} + 1, \dots, s_{i1} + \dots + s_{ik} + \frac{n-h}{2}$ and each $i \geq i_0$. By Theorem 3, it is not hard to see that $T' = T(v_1, \dots, v_k; s_1, \dots, s_k)$ satisfies in (3), for every integers $s_1, \dots, s_k \geq t$, where $t = \max\{s_{i_01}, \dots, s_{i_0k}\}$. Therefore, T and hence T_{i_0} have the form $S(r_1, \dots, r_k)$, for some nonnegative integers r_1, \dots, r_k . This contradicts Theorem 12. \square

4 Integral trees with nullity 2 and 3

In this section, we characterize integral trees with nullity 2 and 3. In order to do this in a simple manner, we use the following interesting fact which is called the Parter–Wiener theorem [6, 9]. We recall that integral trees with nullity 0 and 1 are classified in [1, 8].

Theorem 15. *If T is a tree and $\text{mult}(T; \lambda) \geq 2$ for some λ , then there exists $v \in \mathcal{V}(T)$ such that $\text{mult}(T - v; \lambda) = \text{mult}(T; \lambda) + 1$.*

In the next theorem, we generalize an interesting result in [1] by a short and simple proof.

Theorem 16. *Let T be a tree with nullity 1 and no eigenvalue in $(0, 1) \cup (1, 2)$. Then either T is the one-vertex tree or $T = S(p)$, for some $p \geq 1$.*

Proof. Let T be of order n . First, suppose that $\text{mult}(T; 1) \leq 1$. Since the spectrum of eigenvalues of T is symmetric around 0 and the sum of squares of all eigenvalues of T equals $2(n-1)$, we find that $4(n-3) \leq 2(n-1)$, which in turn implies that $n \leq 5$. Among the trees of order at most 5, the one-vertex tree is the only tree satisfying the assumption of the theorem. Next, suppose that $\text{mult}(T; 1) \geq 2$. By Theorem 15, there exists a vertex v such that $\text{mult}(T - v; 1) = \text{mult}(T; 1) + 1$. Therefore, by Theorem 3, we have $m(T - v) = 0$. It follows from Corollary 6 that $T - v$ is a vertex disjoint union of p copies of P_2 , for some $p \geq 1$, yielding the result. \square

With an easy calculation, we get that $\varphi(S(p)) = x(x^2 - p - 3)(x^2 - 1)^{p+1}$. So we reach to the following conclusion from Theorem 16.

Corollary 17. [1] *Every integral tree with nullity 1 is of the form $S(p^2 - 3)$, for some $p \geq 2$.*

Theorem 18. *Let T be a tree with nullity 2 and no eigenvalue in $(0, 1) \cup (1, 2)$. Then either T is the tree of Figure 3 or $T = S(p, q)$, for some nonnegative integers p, q .*

Proof. Let T be of order n . First, suppose that $\text{mult}(T; 1) \leq 1$. With a similar argument given in the proof of Theorem 16, we have $4(n - 4) \leq 2(n - 1)$, which in turn implies that $n \leq 7$. Among the trees of order at most 7, the only tree satisfying the assumption of the theorem is the tree depicted in Figure 3. Next, suppose that $\text{mult}(T; 1) \geq 2$. By Theorem 15, there exists a vertex v such that $\text{mult}(T - v; 1) = \text{mult}(T; 1) + 1$. Therefore, by Theorem 3, $T - v$ has nullity 1 and has no eigenvalue in $(0, 1) \cup (1, 2)$. In view of Theorem 16, $T - v$ has one of the form $rK_2 \cup K_1$ or $pK_2 \cup S(q)$, for some nonnegative integers r, p, q . If the former occurs, then T would have a perfect matching, which contradicts Corollary 2. Hence the latter is the case. If the neighbor of v in $S(q)$ is not a vertex of degree 2, then again T would have a perfect matching, a contradiction. So v is adjacent with a vertex of degree 2 in $S(q)$ and thus $T = S(p, q)$, for some nonnegative integers p, q . \square

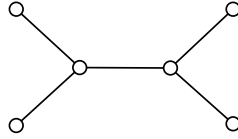


Figure 3: The unique integral tree with nullity 2.

By Lemma 12, no tree of the form $S(p, q)$ is integral. This together with Theorem 18 imply the following.

Corollary 19. *There is only one integral tree with nullity 2; namely, the tree depicted in Figure 3.*

Theorem 20. *The star of order 5 is the only integral tree with nullity 3.*

Proof. Let T be an integral tree of order n with nullity 3. Assume that $\text{mult}(T; 1) \leq 1$. With a similar argument given in the proof of Theorem 16, we have $4(n - 5) \leq 2(n - 1)$ and hence $n \leq 9$. Among the trees of order at most 9, there is only one integral tree with nullity 3 that is the star of order 5, we are done. Towards a contradiction, suppose that $\text{mult}(T; 1) \geq 2$. By Theorem 15, there exists a vertex v such that $\text{mult}(T - v; 1) = \text{mult}(T; 1) + 1$. Moreover, by Theorem 3, $T - v$ has nullity 2 and has no eigenvalue in $(0, 1) \cup (1, 2)$. It easily follows from Lemma 1 that $T - v$ has no isolated vertex. From Theorems 16 and 18, it follows for some nonnegative integers r, p, q that $T - v$ is of one of the following forms:

- (i) $rK_2 \cup S(p) \cup S(q)$;

(ii) $rK_2 \cup S(p, q)$;

(iii) $rK_2 \cup Y$, where Y is the tree depicted in Figure 3.

If (i) is the case, then by Corollary 2, v is necessarily adjacent to two vertices of degree 2 in $S(p)$ and $S(q)$. This means that $T = S(p, r, q)$, which contradicts Theorem 12.

In the case (ii), using Corollary 2, v is adjacent either to a vertex of degree 2 in $S(p, q)$ or to the common neighbor of the two central vertices of $S(p, q)$. First, suppose that the former occurs. If $r \geq 1$, then $T = S(r-1, p, q)$, which again contradicts Theorem 12. In the case $r = 0$, by (1), we find that

$$\varphi(T) = x^3(x^2 - 1)^{p+q}((x^2 - 2)(x^2 - p - 3)(x^2 - q - 3) - 2x^2 + q + 5).$$

Since one can easily check that the polynomial $(x - 2)(x - p - 3)(x - q - 3) - 2x + q + 5$ has a zero in $(1, 2)$, we get a contradiction. Next, suppose that the latter occurs. Applying (1),

$$\varphi(T) = x^3(x^2 - 1)^{a+b+c-8}((x^2 - a)(x^2 - b)(x^2 - c) - 3x^2 + a + b + c - 2),$$

where $a = p + 3$, $b = q + 3$, and $c = r + 2$. By symmetry, we may assume that $a \geq b \geq c$. Letting $g(x) = (x - a)(x - b)(x - c) - 3x + a + b + c - 2$, we have

$$\begin{aligned} g(a) &= -(2a - b - c + 2), \\ g(a + 1) &= (a - b)(a - c) - 4, \\ g(a + 2) &= 2(a - b)(a - c) + 3(2a - b - c). \end{aligned}$$

It follows that g has a zero in $(a, a + 1) \cup (a + 1, a + 2)$ unless $a = b + 2 = c + 2$ in which case $g(x) = (x - a - 1)(x - a + 2)(x - a + 3)$. So we are done as $a - 2$ and $a - 3$ cannot be both perfect squares, since in this case $a \geq 5$.

For the case (iii), using Corollary 2, v is necessarily adjacent to one of the two vertices of degree 3 in Y . It is easily seen that $\varphi(T) = x^3(x^2 - 1)^r(x^4 - (r + 6)x^2 + 4r + 6)$ has a zero in $(1, 2)$, a contradiction. The proof is now complete. \square

Finally, we mention that one can apply a similar method to find all integral trees with other small nullities. By [1], among trees up to fifty vertices, there is no integral tree with nullity 4 and there are two integral trees with nullity 5.

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